Calculus 1
Workbook Solutions

Intermediate value theorem

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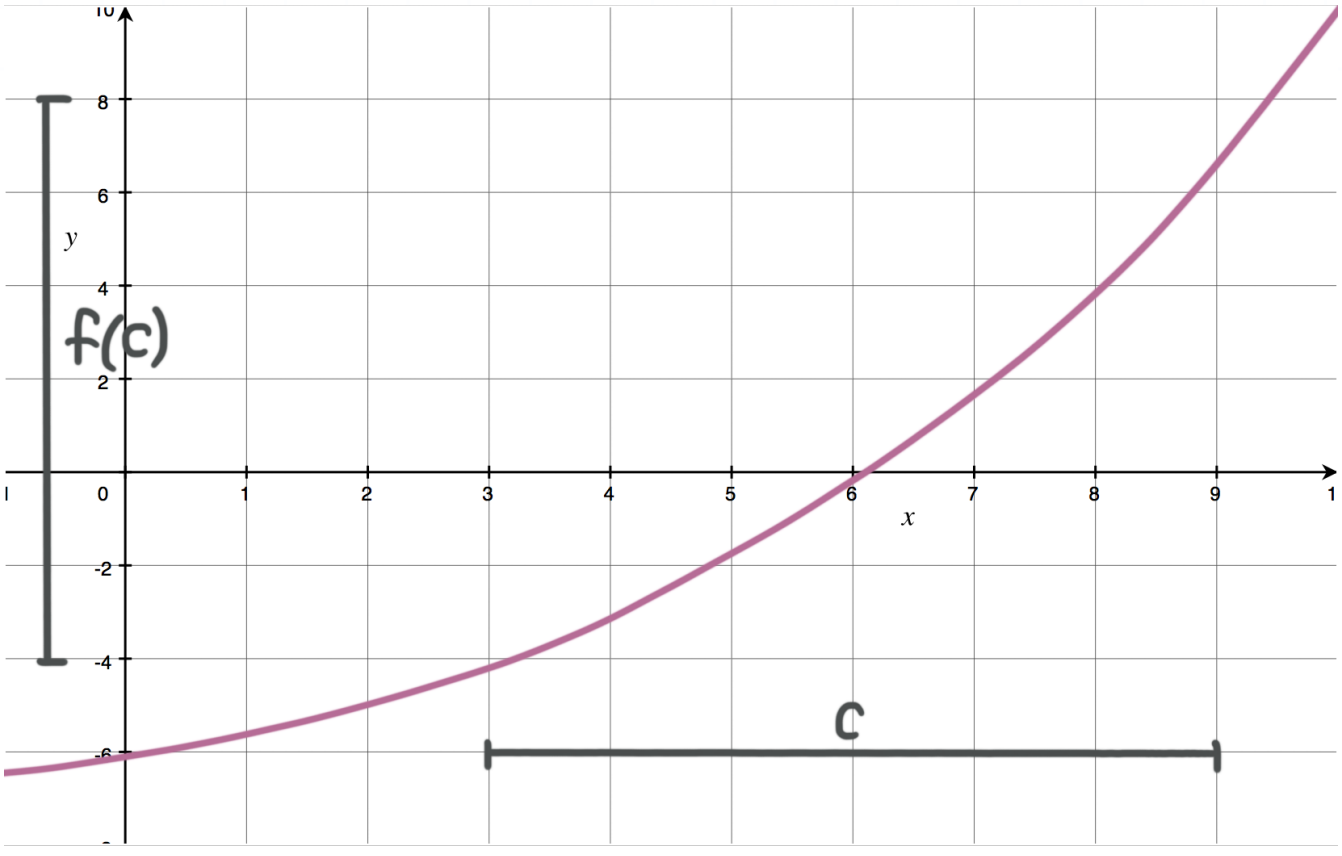
MATH

INTERMEDIATE VALUE THEOREM

■ 1. Suppose the function $f(x)$ is continuous on the closed interval $[3,9]$ and that $f(3) = -4$ and $f(9) = 8$. The Intermediate Value Theorem guarantees that for some value c in the interval $[3,9]$, $f(c)$ will have a value in what interval?

Solution:

The function will have a root on the interval $[-4,8]$. Under the conditions of the Intermediate Value Theorem, the domain of c is the interval $[3,9]$ and the range of $f(c)$ is $[-4,8]$.



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■ 2. Suppose the function $g(x)$ is continuous on the closed interval $[0,12]$, and that $f(0) = 12$ and $f(12) = 2$. What is the domain of some value c if the Intermediate Value Theorem guarantees that for that value c, $f(c)$ will have a range of $[2,12]$?

Solution:

Under the conditions of the Intermediate Value Theorem (IVT), the domain of c is the interval in which the values of $g(x)$ are used in the question. Thus the domain of c to meet the conditions of the IVT is $[0,12]$ and the range of $f(c)$ is $[2,12]$, as shown in the graph of a hypothetical function $g(x)$ below.

■ 3. Suppose the function $h(x)$ is being discussed with some value c in regards to the Intermediate Value Theorem on the interval from a to b,

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where $b > a$. Select which conditions must be satisfied for $h(x)$ and the interval to apply to the Intermediate Value Theorem.

I - $f(c)$ must satisfy $f(a) \leq f(c) \leq f(b)$

II - $h(x)$ must be continuous and differentiable

III – The number c is $a < c < b$

IV - $h(x)$ must be continuous

V – The interval from a to b is an open interval

VI – The number c is $a \leq c \leq b$

VII – The interval from a to b is a closed interval

Solution:

I, IV, and VII must be true. The Intermediate Value Theorem states:

> A function $y = f(x)$ is continuous on a closed interval $[a, b]$ and takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.

Statements I, IV, and VII are conditions. Statement VI is a conclusion of the IVT. Statements II, III, and V are erroneous. This situation is modeled in the figure below.

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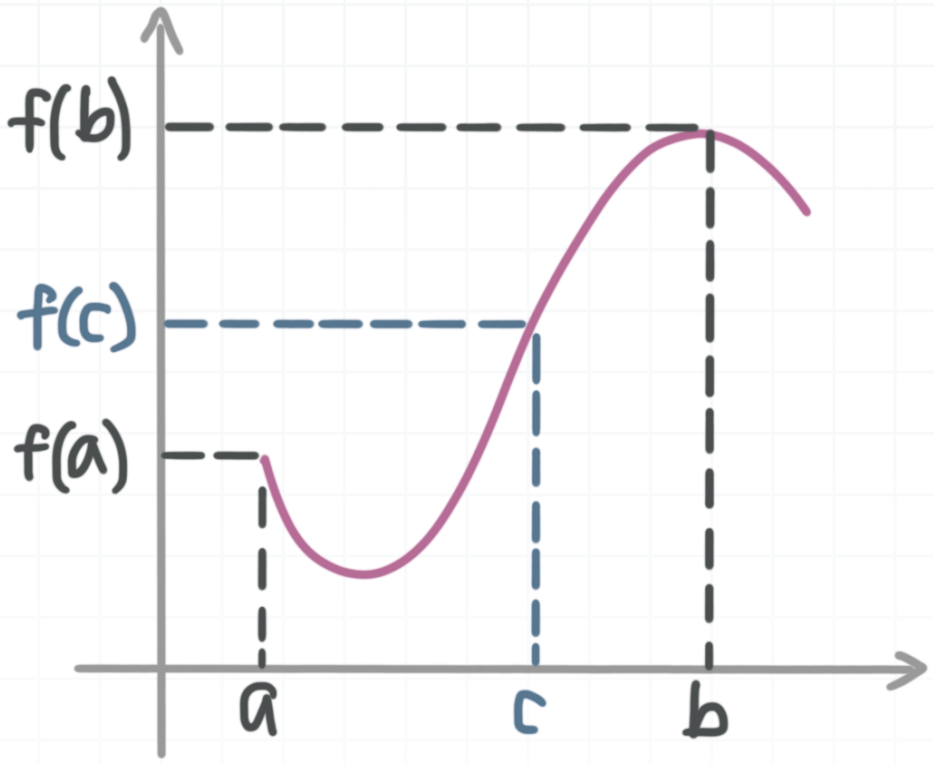

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INTERMEDIATE VALUE THEOREM WITH AN INTERVAL

■ 1. The value $c = -1$ satisfies the conditions of the Intermediate Value Theorem for the function on the interval $[-3,5]$ because $f(c)$ equals what value?

$$f(x) = \frac{1}{4}(2x+5)(x-3)^2$$

Solution:

The Intermediate Value Theorem (IVT) states that a function $y = f(x)$ is continuous on a closed interval $[a, b]$ and takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$. In this problem, $f(a) = f(-3) = -9$ and $f(b) = f(5) = 15$. Then,

$$f(c) = f(-1) = \frac{1}{4}\big(2(-1)+5\big)(-1-3)^2 = 12$$

The IVT requires that $f(a) \le f(c) \le f(b)$ and $-9 \le 12 \le 15$.

■ 2. The value $c = 2$ does not satisfy the conditions of the Intermediate Value Theorem for $g(x) = 2x^2 - 11x + 4$ on the interval $[-2,8]$ because $g(c)$ equals what value?

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Solution:

The Intermediate Value Theorem (IVT) states that a function $y = f(x)$ is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$. In this problem, $g(a) = g(-2) = 34$ and $g(b) = g(8) = 44$. However, $g(c) = g(2) = 2(2)^2 - 11(2) + 4 = -10$. The IVT requires that $f(a) \leq f(c) \leq f(b)$, but -10 is not between 34 and 44.

■ 3. What value of c is guaranteed by the Intermediate Value Theorem on the interval $[-3,3]$ if $h(x) = 3(x + 1)^3$ and $h(c) = 24$?

Solution:

The Intermediate Value Theorem (IVT) states that a function $y = f(x)$ is continuous on a closed interval $[a, b]$ and takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$. In this problem, $h(a) = h(-3) = -24$ and $h(b) = h(3) = 192$. Thus, if $h(c) = 24$, the IVT requires that since $f(a) \leq f(c) \leq f(b)$, $a \leq c \leq b$. Thus, since $h(c) = 24$, we get

$$3(c + 1)^3 = 24$$

$$(c + 1)^3 = 8$$

$$c + 1 = 2$$

$$c = 1$$

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■ 4. What value of c is guaranteed by the Intermediate Value Theorem on the interval $[-5,6]$ if $f(c) = -6$ and

$$f(x) = \begin{cases} 3x - 10 & \text{if } x \leq 0 \\ x^2 + 3x - 10 & \text{if } 0 < x < 2 \\ 3x - 6 & \text{if } x \geq 2 \end{cases}$$

Solution:

The Intermediate Value Theorem (IVT) states that a function $y = f(x)$ is continuous on a closed interval $[a, b]$ and takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$. In this problem, first confirm that the function $f(x)$ is continuous on the interval $[-5,6]$ by evaluating the function on both sides of $x = 0$ and $x = 2$. The function is continuous, as shown in the graph below.

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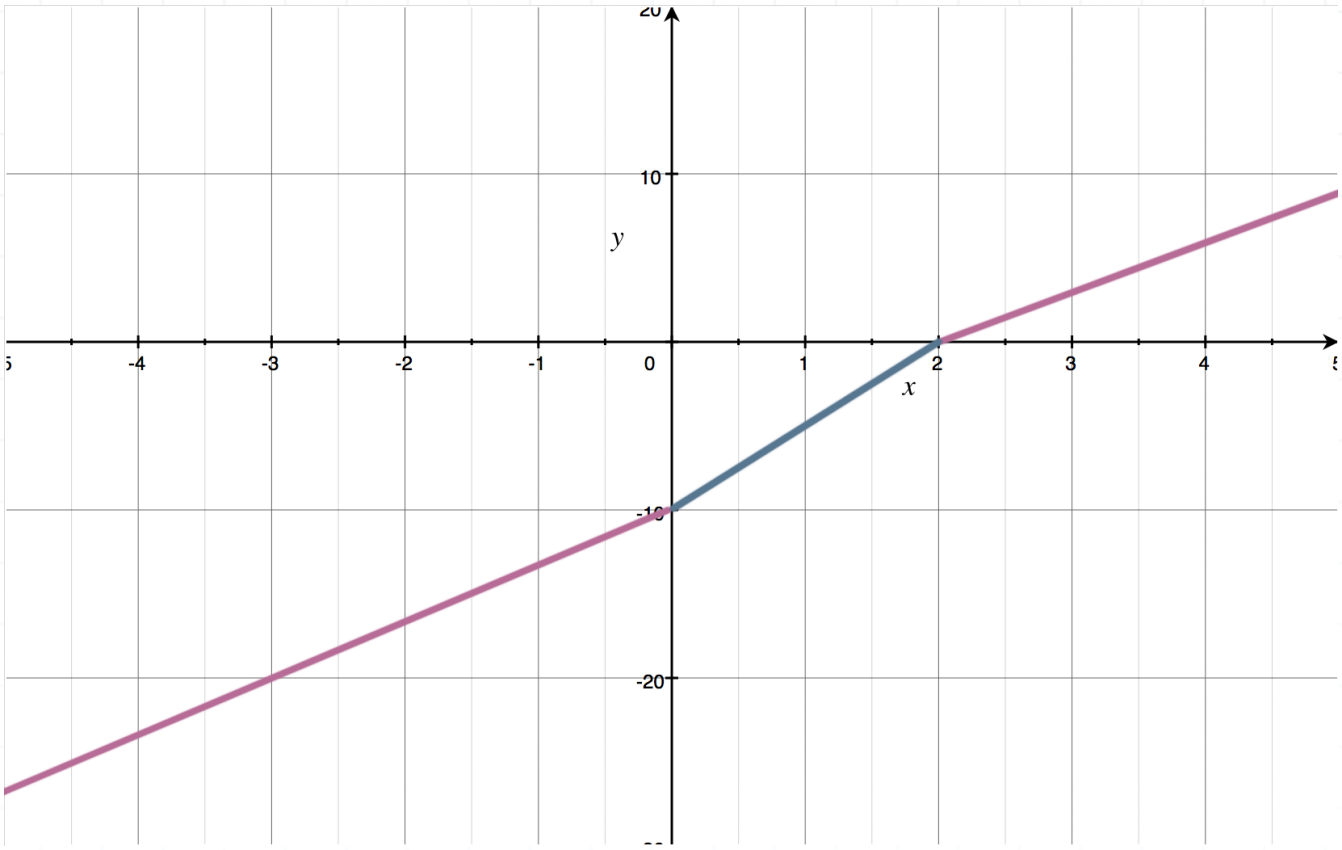

$f(a) = f(-5) = -25$ and $f(b) = f(6) = 12$. Thus, if $f(c) = -6$, the IVT requires that since $f(a) \leq f(c) \leq f(b)$, $a \leq c \leq b$. Thus, since $f(c) = -6$,

$$c^2 + 3c - 10 = -6$$

$$c^2 + 3c - 4 = 0$$

$$(c + 4)(c - 1) = 0$$

$$c = 1$$

when using $x^2 + 3x - 10$. Because $f(x)$ is defined piecewise, there can be other values of c that might satisfy the IVT, but there are no other values of c that satisfy the conditions.

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■ 5. The value $c = 5$ satisfies the conditions of the Intermediate Value Theorem for the function on the interval $[3,9]$ because $g(c)$ equals what value?

$$g(x) = \frac{x^2 - 9}{x + 3}$$

Solution:

The Intermediate Value Theorem (IVT) states that a function $y = f(x)$ is continuous on a closed interval $[a, b]$ and takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$. In this problem, $g(a) = g(3) = 0$ and $g(b) = g(9) = 6$. Then,

$$g(c) = g(5) = \frac{5^2 - 9}{5 + 3} = 2$$

The IVT requires that $g(a) \leq g(c) \leq g(b)$ and $0 \leq 2 \leq 6$. Although $g(x)$, as defined, contains a discontinuity at $x = -3$, the function is continuous in the given interval, therefore satisfying the IVT.

■ 6. What value of c is guaranteed by the Intermediate Value Theorem on the interval $[3,6]$ if c is a root of $h(x)$.

$$h(x) = \frac{x^3 - 4x^2 - 11x + 30}{x^2 - 4}$$

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Solution:

The Intermediate Value Theorem (IVT) states that a function $y = f(x)$ is continuous on a closed interval $[a, b]$ and takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.

In this problem, $h(a) = h(3) = -12/5$ and $h(b) = h(6) = 9/8$. Thus, if $h(c)$ is a root of $h(x)$, then $h(c) = 0$. The IVT requires that since $f(a) \leq f(c) \leq f(b)$, $a \leq c \leq b$. Thus, since $h(c) = 0$, then

$$\frac{c^3 - 4c^2 - 11c + 30}{c^2 - 4} = 0$$

Solving this equation gives $c = 5$. Note that although $h(x)$, as defined, contains discontinuities at $x = -2$ and $x = 2$, the function is continuous in the given interval, therefore satisfying the IVT.

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INTERMEDIATE VALUE THEOREM WITHOUT AN INTERVAL

■ 1. Use the Intermediate Value Theorem to prove that the equation $2e^x = 3\cos x$ has at least one positive solution. In what interval is that solution?

Solution:

Let $f(x) = 2e^x - 3\cos x$. The root of $f(x)$ is a solution to the given equation. The Intermediate Value Theorem guarantees that the function $f(x)$ has a root in a certain closed interval if the function's value changes sign in that closed interval.

Consider the interval $[0,1]$. Then,

$$f(0) = 2e^0 - 3\cos 0 = 2 - 3 = -1$$

$$f(1) = 2e^1 - 3\cos 1 = 2e - 1.6209$$

which is approximately 3.8157. Since the function's value changes sign in the interval $[0,1]$, and since $f(x)$ is continuous in the interval, by the Intermediate Value Theorem the function has a zero in that interval.

■ 2. Use the Intermediate Value Theorem to prove that the equation $3\sin x + 7 = x^2 - 2x - 2$ has at least one positive solution. In what interval is that solution?

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Solution:

Let $g(x) = 3\sin x + 7 - (x^2 - 2x - 2)$. The root of $g(x)$ is a solution to the given equation. The Intermediate Value Theorem guarantees that the function $g(x)$ has a root in a certain closed interval if the function's value changes sign in that closed interval.

Consider the interval $[3,4]$. Then,

$$g(3) = 3\sin(3) + 7 - \left(3^2 - 2(3) - 2\right) = 6.4234$$

$$g(4) = 3\sin(4) + 7 - \left(4^2 - 2(3) - 2\right) = -3.2704$$

Since the function's value changes sign in the interval $[3,4]$, and $g(x)$ is continuous on the interval, the function has a zero in that interval.

■ 3. Use the Intermediate Value Theorem to prove that the equation $x^6 - 9x^4 + 7 = x^5 - 8x^3 - 9$ has at least one positive solution. In what interval is that solution?

Solution:

Let $h(x) = (x^6 - 9x^4 + 7) - (x^5 - 8x^3 - 9)$. The root of $h(x)$ is a solution to the given equation. The Intermediate Value Theorem guarantees that the function $h(x)$ has a root in a certain closed interval if the function's value of changes sign in that closed interval.

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Consider the interval $[1,2]$. Then,

$$h(1) = \left((1)^6 - 9(1)^4 + 7\right) - \left((1)^5 - 8(1)^3 - 9\right) = 15$$

$$h(2) = \left((2)^6 - 9(2)^4 + 7\right) - \left((2)^5 - 8(2)^3 - 9\right) = -32$$

Since the function's value changes sign in the interval $[1,2]$, and $h(x)$ is continuous on the interval, by the Intermediate Value Theorem the function has a zero in that interval.

■ 4. Use the Intermediate Value Theorem to prove that the equation $4e^{x-3} = -2(x^3 - 5x + 9)$ has at least one negative solution. In what interval is that solution?

Solution:

Let $f(x) = 4e^{x-3} - 2(x^3 - 5x + 9)$. The root of $f(x)$ is a solution to the given equation. The Intermediate Value Theorem guarantees that the function $f(x)$ has a root in a certain closed interval if the function's value changes sign in that closed interval.

Consider the interval $[-3, -2]$. Then,

$$f(-3) = 4e^{-3-3} - 2\left((-3)^3 - 5(-3) + 9\right) = 6.0099$$

$$f(-2) = 4e^{-2-3} - 2\left((-2)^3 - 5(-2) + 9\right) = -21.97$$

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Since the function's value changes sign in the interval $[-3, -2]$, and $f(x)$ is continuous on the interval, by the Intermediate Value Theorem the function has a zero in that interval.

■ 5. Use the Intermediate Value Theorem to show that the equation has at least one positive solution. In what interval is that solution?

$$6e^{-x} = -\left(\frac{1}{5}x^2 - 4x + 9\right)$$

Solution:

Let

$$g(x) = 6e^{-x} + \frac{1}{5}x^2 - 4x + 9$$

The root of $g(x)$ is a solution to the given equation. The Intermediate Value Theorem guarantees that the function $g(x)$ has a root in a certain closed interval if the function's value changes sign in that closed interval.

Consider the interval $[2,3]$. Then,

$$g(2) = 6e^{-2} + \frac{1}{5}(2)^2 - 4(2) + 9 = 2.612$$

$$g(3) = 6e^{-3} + \frac{1}{5}(3)^2 - 4(3) + 9 = -0.9013$$

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Since the function's value changes sign in the interval $[2,3]$, and $g(x)$ is continuous on the interval, by the Intermediate Value Theorem the function has a zero in that interval.

■ 6. Use the Intermediate Value Theorem to show that the equation $2\sin(4x-1)=\cos(2x-3)$ has at least one negative solution. In what interval is that solution?

Solution:

Let $h(x)=2\sin(4x-1)-\cos(2x-3)$. The root of $h(x)$ is a solution to the given equation. The Intermediate Value Theorem guarantees that the function $h(x)$ has a root in a certain closed interval if the function's value changes sign in that closed interval.

Consider the interval $[-2,-1]$. Then,

$$h(-2)=2\sin\big(4(-2)-1\big)-\cos\big(2(-2)-3\big)=-1.578$$

$$h(-1)=2\sin\big(4(-1)-1\big)-\cos\big(2(-1)-3\big)=1.634$$

Since the function's value changes sign in the interval $[1,2]$, and $h(x)$ is continuous on the interval, by the Intermediate Value Theorem the function has a zero in that interval.

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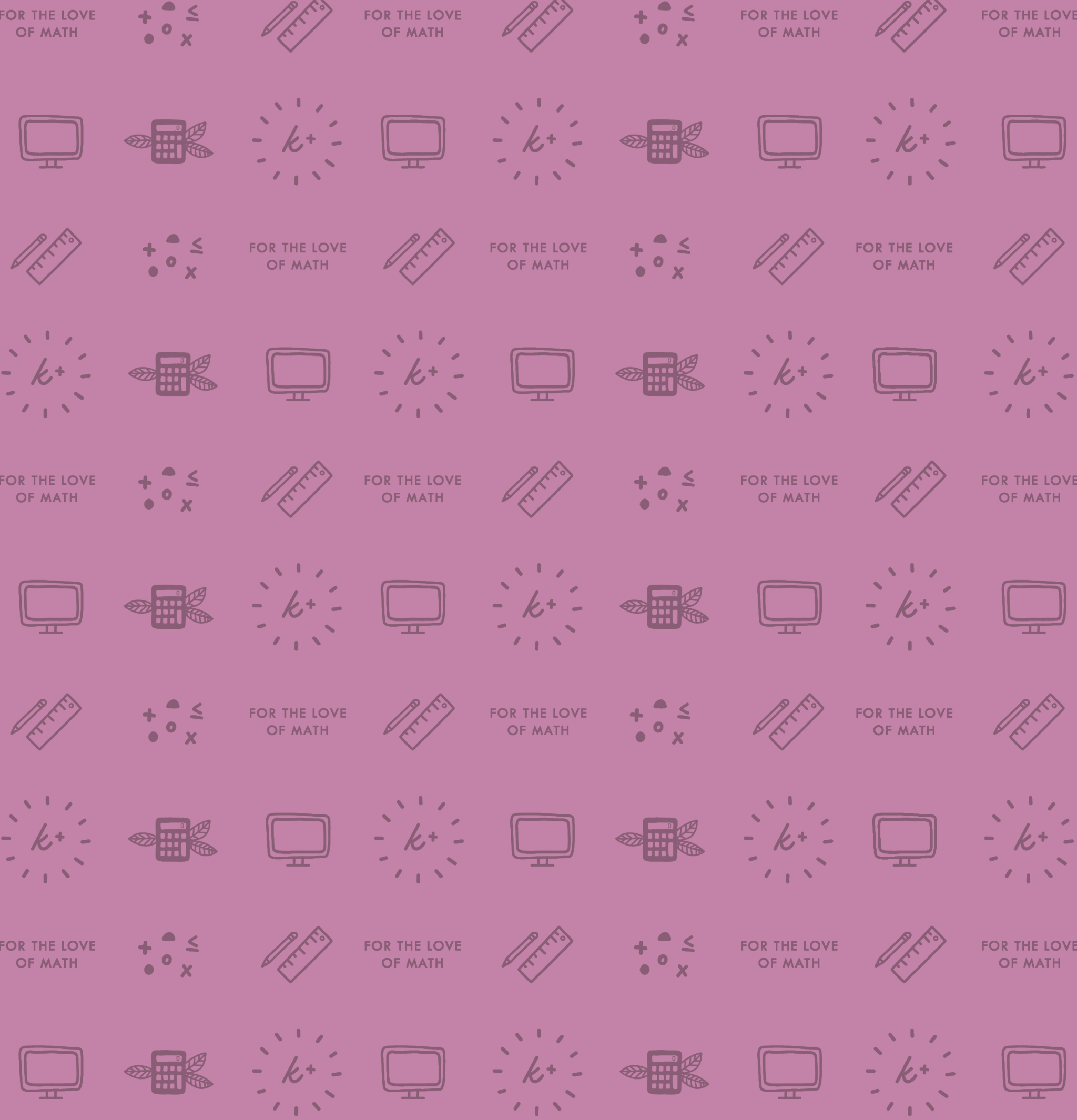

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